Geometric properties of sheaves of coinvariants and conformal blocks

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It work with
Gibney Tarasca

Geom: with Gibney

What are we going to see today?

Describe geometric properties of sheaves over $\overline{M}_{g,n}$ attached to V-modules $W^1, \ldots, W^n$ for V a vertex operator algebra of CohFT-type.

curves of genus g and n-marked points

$C, P_1 \ldots P_n \quad V; W^1 \ldots W^n \rightsquigarrow \mathbb{V}_{C,P_\bullet}(W^1 \ldots W^n)$

① definition (setting)

② Geom: GLOBAL GENERATION

What are Vertex Operator Algebras of CohFT-type?

$V = \bigoplus_{i \geq 0} V_i$ graded vector space

$\mathbf{1} \in V_0$ vacuum vector

$\omega \in V_2$ conformal vector

$Y : V \to \mathrm{End}(V)[[z^{\pm 1}]]$

$A \mapsto \sum_{n \in \mathbb{Z}} A_{(n)} z^{-n-1} = Y(A)$

state-field correspondence

$Y(A,B) \in V((z))$

- $V_0 = \mathbf{1}\mathbb{C}$
- $\dim(V_i) < \infty$
- $\{\omega_{(n)}, c_V \mathrm{Id}_V\} \cong \mathrm{Vir}$ — c_V = central charge
- V is C_2 cofinite — $V/C_2(V)$ finite, $C_2(V) = A_{(-2)}B$
- V is rational → cat. of repr. is semisimple

// finitely many simple modules

What do we mean by V-modules?

$$W = \bigoplus_{i \geq 0} W_i \qquad Y^W \colon V \to \mathrm{End}(W)[[z^{\pm 1}]]$$

graded vector space

$$A \mapsto \sum_{n\in\mathbb{Z}} A^W_{(n)} z^{-n-1}$$

state-field correspondence

- $W_0 \neq 0$
- $\dim(W_i) < \infty$
- $\{\omega^W_{(n)}, c_V \mathrm{Id}_W\} \cong \mathrm{Vir}$
- $A^W_{(n)} W_j \subseteq W_{j+\deg(A)-n-1}$
- when W is simple $L_0(w) = (\deg(w) + a_W)w$

c_V = central charge $\in \mathbb{Q}$

admissibility

$\hookrightarrow$ conf. dim. $\in \mathbb{Q}$

Construction of $\mathbb{V}_{(C,P)}(V;W)$ } space of coinvariants

Main idea: $\mathbb{V}_{(C,P)}(V;W)$ is a quotient of W by a Lie algebra encoding the geometrical input.

Lie Algebras

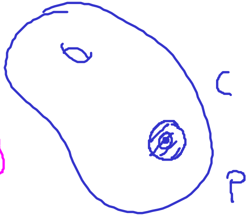
$V = L_\ell(\mathfrak{g})$

$\ell \in \mathbb{Z}_{\geq 1}$

$$\frac{W}{(\mathfrak{g}\otimes \mathcal{O}_{C\setminus P})\odot W}$$

map of Lie alg

W module for $\hat{\mathfrak{g}} = \mathfrak{g}((t_P)) \oplus \ell\mathbb{C}$



Construction of $\mathbb{V}_{(C,P)}(V;W)$

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[A] Construct a Lie algebra $\mathcal{L}_{C\setminus P}(V)$ acting on W.

$\mathcal{V} \to C$

$v \xrightarrow{\nabla} v \otimes \omega$

$\mathcal{L}_{C\setminus P}(V) = H^0\left(C\setminus P, \frac{\mathcal{V}\otimes\omega}{\nabla\mathcal{V}}\right)$

↓ take its expansion at P

$\mathcal{L}_P(V) = H^0\left(D_P^\times, \frac{\mathcal{V}\otimes\omega}{\nabla\mathcal{V}}\right)$

$\cong t_P$

$[A t_P^n] * w = A^W_{(n)}(w)$

$W \circlearrowleft \frac{V((t_P))}{\nabla = L_{-1}\oplus\frac{d}{dt_P}} = L_{t_P}(V)$

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[A] Construct a Lie algebra $\mathcal{L}_{C\setminus P}(V)$ acting on W.
[B] Define the *space of coinvariants*

$$\mathbb{V}_{(C,P,t_P)}(V; W) := \frac{W}{\mathcal{L}_{C\setminus P}(V)(W)}.$$

local definition

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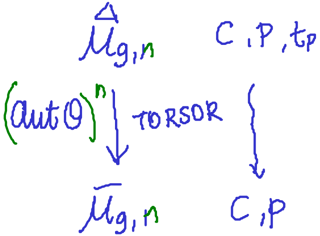
$$\mathbb{V}_{(C,P,t_P)}(V;W) := \frac{W}{\mathcal{L}_{C\setminus P}(V)W}.$$

[C] Forget the coordinate t_P and obtain $\mathbb{V}_{(C,P)}(V;W)$.

→ descent



Some first properties [D - Gibney - Tarasca]

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- The spaces $\mathbb{V}_{(C,P_\bullet)}(V; W^1, \ldots, W^n)$ fit together to define the sheaf $\mathbb{V}_g(V; W^1, \ldots, W^n)$ over $\overline{M}_{g,n}$, which is actually a vector bundle of finite rank.

locally free

finite dim

det($\mathbb{V}$)

line bundles!

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  - $\mathbb{V}_{(C,P_\bullet)}(V; W^1, \ldots, W^n)$ are finite dimensional. C2-cofiniteness

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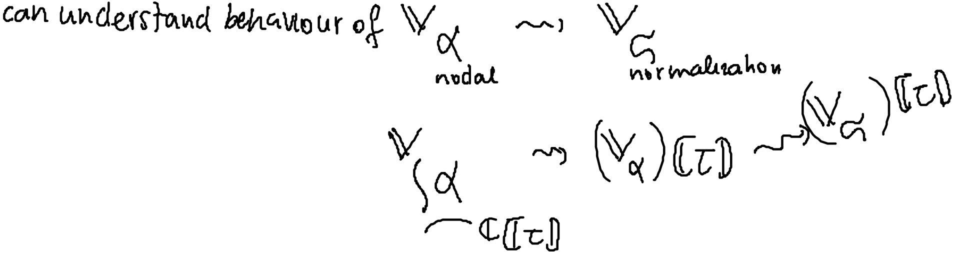
  - $\mathbb{V}_{(C,P_\bullet)}(V; W^1, \dots, W^n)$ are finite dimensional.
  - Factorization rules and the sewing theorem hold.



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  - Factorization rules and the sewing theorem hold.
  - C $\mathbb{V}_g(V; W^1, \ldots, W^n)$ is equipped with a twisted D-module structure with logarithmic singularities along the boundary.

A+C guar. $\mathbb{V}_g$ are v.b. of finite rank over $\overline{M}_{g,n}$
↳ + B

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  - $\mathbb{V}_g(V; W^1, \ldots, W^n)$ is equipped with a twisted D-module structure with logarithmic singularities along the boundary.
- The assignment $(g, W^1, \ldots, W^n) \mapsto \mathrm{Ch}(\mathbb{V}_g(V; W^1, \ldots, W^n))$ defines a semisimple CohFT.

$L_\ell(\mathfrak{g})$

$\ell \geq 1$

- Relation of C.B with Θ-function
- $\mathbb{V}_0(L_\ell(\mathfrak{g}); W^1 - W^n)$ is globally generated

Global generation

Global generation

A vector bundle $\mathbb{V}$ on a variety M is globally generated if there exists a vector space P and a surjective map

CONSTANT sheaf

$$\underbrace{P \otimes \mathcal{O}_M}_{} \longrightarrow\!\!\!\!\rightarrow \mathbb{V}$$

If $\mathbb{V}$ globally gen, $\det(\mathbb{V})$ defines maps from $M \longrightarrow \mathbb{P}^N$

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Theorem [Fakhruddin] The sheaves of coinvariants $\mathbb{V}_0(L_\ell(\mathfrak{g}); W^1, \ldots, W^n)$ are a quotient of $(W_0^1 \otimes \cdots \otimes W_0^n) \otimes \mathcal{O}_{\overline{M}_{0,n}}$, hence they are globally generated.

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Theorem [D - Gibney] The sheaves of coinvariants $\mathbb{V}_0(V; W^1, \dots, W^n)$ are a quotient of $(W^1_0 \otimes \cdots \otimes W^n_0) \otimes \mathcal{O}_{\overline{M}_{0,n}}$, hence they are globally generated **if V is strongly generated in degree 1**.

$$V = \langle A^{j_1}_{(-j_1)} \cdot A^{j_2}_{(-j_2)} \cdots A^{j_n}_{(-j_n)} \mathbb{1} \mid A^{j} \in V_1, \ j_n \in \mathbb{Z} \rangle_{\mathbb{C}}$$

First example

$V = L(\frac{1}{2}, 0)$ so that $\mathrm{Rep}(V) = \{L(\frac{1}{2}, 0), L(\frac{1}{2}, \frac{1}{2}), L(\frac{1}{2}, \frac{1}{16})\} = \{V, W(\frac{1}{2}), W(\frac{1}{16})\}$.

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Over $\overline{M}_{0,4}$ there are only three sheaves with degree different from zero:

$\parallel$

$\mathbb{P}^1$

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$\mathbb{V}_0(V; W(\frac{1}{2}), W(\frac{1}{2}), W(\frac{1}{16}), W(\frac{1}{16}))$	1	1
$\mathbb{V}_0(V; W(\frac{1}{16}), W(\frac{1}{16}), W(\frac{1}{16}), W(\frac{1}{16}))$	2	-1

not G.G.!!!

Second example

$V = V_L$ with $L = \mathbb{Z}\epsilon$ lattice with $(\epsilon, \epsilon) = 2 \cdot 4$, so that $\mathrm{Rep}(V_L) \cong \mathbb{Z}/8\mathbb{Z} = \{W_i\}_{i=0}^{7}$.

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$$c_1(\mathbb{V}_0(2,2,2,2)) = \left(\frac{4}{16}\psi_1 + \frac{4}{16}\psi_2 + \frac{4}{16}\psi_3 + \frac{4}{16}\psi_4\right) - \left(\frac{16}{16}\delta_{[2,2][2,2]} + \frac{16}{16}\delta_{[2,2][2,2]} + \frac{16}{16}\delta_{[2,2][2,2]}\right).$$

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$2 \cdot 2 \cdot k$

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$W_k \; W_k \; W_k \; W_k$

conf dim of W_2

conf. dim of W_4

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$$\deg(\mathbb{V}_0(2,2,2,2)) = \left(\frac{4}{16}+\frac{4}{16}+\frac{4}{16}+\frac{4}{16}\right) - \left(\frac{16}{16}+\frac{16}{16}+\frac{16}{16}\right) = 1 - 3 = -2.$$

$-k$

$\dim(W_2)_0 > 1$

Comments on the proof — general setting

Theorem [D - Gibney] If V is strongly generated in degree 1, then the sheaves of coinvariants $\mathbb{V}_0(V; W^1, \ldots, W^n)$ are a quotient of $(W^1_0 \otimes \cdots \otimes W^n_0) \otimes \mathcal{O}_{\overline{M}_{0,n}}$, hence they are globally generated.

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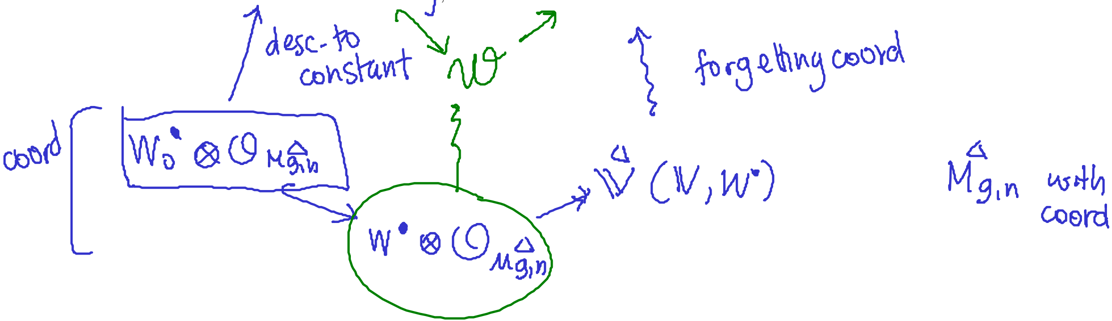
1. The map $(W_0^1 \otimes \cdots \otimes W_0^n) \otimes \mathcal{O}_{\overline{M}_{0,n}} \to \mathbb{V}_g(V; W^1, \ldots, W^n)$ is always defined.



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2. Enough to show that

$$W_0^1 \otimes \cdots \otimes W_0^n \hookrightarrow W^1 \otimes \cdots \otimes W^n \to \mathbb{V}_{(C,P_\bullet,t_\bullet)}(V; W^1, \ldots, W^n)$$

is surjective for every curve C of genus g.

Global map

check surj LOCAL.

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3. We need to show that: "induction on degree"

$$\underbrace{w^1 \otimes \cdots \otimes w^n}_{\deg = d} \in W^\bullet = \underbrace{\quad}_{\deg < d} + \mathcal{L}_{C \setminus P}(V) \cdot W^\bullet$$

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A. **strongly generated in degree 1:** we can assume that $w = w^1 \otimes \cdots \otimes w^n$ has one component

$$w^i = A_{(-j)} u^i \qquad\qquad u^i \in W^1_{\deg(w^i)-j} \quad A \in V_1 \text{ and } j \geq 1$$

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B. **g=0:** we can construct an element $\sigma \in \mathcal{L}_{C \setminus P_\bullet}(V)$ such that

- $\sigma_{P_i} = A_{-j} +$ lower degree terms
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Conclusion

$$\deg\left(w - \sigma(w^1 \otimes \cdots \otimes u^i \otimes \cdots w^n)\right) \leq \deg(w) - 1$$

Concluding comments

Can we identify exactly when coinvariants are globally generated and when they are not?

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Quasi-Theorem [D - Gibney] Under the assumptions

- $\dim(W_0^i) = 1$ for all i
- $\operatorname{rank}(\mathbb{V}_0(V; W^1, \ldots, W^n)) = 1$,

then $\mathbb{V}_0(V; W^1, \ldots, W^n)$ is globally generated over $\overline{M}_{0,n}$.

$W_0^1 \otimes \cdots \otimes W_0^n \to \mathbb{V}(M_0)$

P

Zhu-alg is comm ✓

Thank you!